

Accurate energy spectrum for the quantum Yang-Mills mechanics with nonlinear color oscillations

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May 19, 2014

Abstract

Yang-Mills theory as the foundation for quantum chromodynamics is a non-Abelian gauge theory with self-interactions between vector particles. Here, we study the Yang-Mills Hamiltonian with nonlinear color oscillations in the absence of external sources corresponding to the group $SU(2)$. In the quantum domain, we diagonalize the Hamiltonian using the optimized trigonometric basis expansion method and find accurate energy eigenvalues and eigenfunctions for one and two degrees of freedom. We also compare our results with the semiclassical solutions.

PACS numbers: 11.10.Lm, 03.65.Ge

1 Introduction

All important theories of modern physics (except gravitation) such as quantum electrodynamics, Weinberg-Salam electroweak theory, the standard model of particle physics, and the grand unified theories are quantized versions of the Yang-Mills theory. After quantization, the quanta of the fields are usually interpreted as particles. The Yang-Mills theory is a very active field of research as a gauge theory based on the $SU(N)$ group. These theories have many subtle and interesting properties which are still under several investigations from physical or mathematical point of views. In physics, the Yang-Mills theory is important since the predictions of the Standard Model agree with the experiments with amazing accuracy. In the context of mathematics, the Yang-Mills theory inspires various important ideas in algebra, analysis, and geometry. Indeed, the quantum Yang-Mills theories can be considered as a limit of a more fundamental string theory. Note that the mass gap problem which is discovered by physicists from experiment is one of the unsolved millennium problems [1].

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There has been much attention in the literature for solving classical Yang-Mills equations and obtaining non-perturbative effects related with the ground state of quantum chromodynamics [2, 3]. In Ref. [4] the classical Yang-Mills equations without external sources are studied to address the problem of asymptotic states of the theory and the structure of the vacuum. In classical domain, the solutions exhibit nonlinear oscillations of the color degrees of freedom similar to the massive nonlinear plane waves in classical electrodynamics [5].

Here, we study this problem in the quantum domain and find the accurate energy eigenvalues and eigenfunctions of the Yang-Mills Hamiltonian describing nonlinear color oscillations. This problem is not exactly solvable and we should resort to a numerical method. The numerical methods for solving the time-independent Schrödinger equations can be categorized into two groups. The first group is based on matrix diagonalization by representing the wave function in terms of a finite orthogonal set of basis functions. Then, the desired eigenvalues and eigenfunctions can be obtained by the diagonalization of a Hamiltonian matrix computed from these basis functions. The second group consists of the iterative methods that are based on repeated numerical integrations of the Schrödinger equation with adjustments of the energy spectrum.

In this paper, following the first category, we use the optimized trigonometric basis expansion method to find the solutions. Thus, we expand the wave functions in terms of the trigonometric basis functions obeying Dirichlet boundary condition. Then we use the Rayleigh-Ritz variational scheme to optimize the domain of the basis functions which leads to highly accurate solutions [6–10].

2 Yang-Mills equations without external sources

Let us consider the Yang-Mills equations in the absence of external sources corresponding to $SU(2)$ in the Minkowski space [4]

$$\partial_\mu G_{\mu\nu}{}^a + g\epsilon^{abc}A_\mu{}^b G_{\mu\nu}{}^c = 0, \quad (1)$$

$$G_{\mu\nu}{}^a = \partial_\mu A_\nu{}^a - \partial_\nu A_\mu{}^a + g\epsilon^{abc}A_\mu{}^b A_\nu{}^c, \quad (2)$$

where Latin indices range from 1 to 3, Greek indices range from 0 to 4, and g is the gauge coupling constant. We look for a solution in a coordinate system so that the Poynting vector $T_{0j} = G_{0i}{}^a G_{ji}{}^a$ vanishes. Here

$$T_{\mu\nu} = -G_{\mu\lambda}{}^a G_{\nu}{}^{\lambda a} + \frac{1}{4} g_{\mu\nu} G_{\lambda\rho}{}^a G^{\lambda\rho a}, \quad (3)$$

is the energy-momentum tensor of the system. In the gauge $\partial_i A_i{}^a = 0$ and $A_0{}^a = 0$ we have

$$\epsilon^{abc} A_i{}^b \dot{A}_i{}^c = 0, \quad (4)$$

$$\ddot{A}_i{}^b - \partial_j G_{ji}{}^a + g \epsilon^{abc} A_j{}^b G_{ji}{}^c = 0, \quad (5)$$

$$\dot{A}_i{}^a G_{ij}{}^a = 0. \quad (6)$$

Using Eqs. (4) and (6) we find

$$\dot{A}_i{}^a (\partial_j A_i{}^a - \partial_i A_j{}^a) = 0. \quad (7)$$

So the sufficient condition to satisfy Eq. (7) is

$$(a) \partial_j A_i{}^a = 0, \quad (b) \dot{A}_i{}^a = 0, \quad (c) \partial_j A_i{}^a - \partial_i A_j{}^a = 0. \quad (8)$$

In the following sections we consider case (a) where $A_i{}^a$ have no spacial dependence.

3 Color space with two degrees of freedom

If we take $\partial_j A_i{}^a = 0$ ($i, j = 1, 2, 3$), the potential depends only on the time and the Yang-Mills equations are given by [5]

$$\ddot{A}_i{}^a + g^2 (A_j{}^b A_j{}^b A_i{}^a - A_j{}^a A_j{}^b A_i{}^b) = 0, \quad (9)$$

where dot denotes differentiation with respect to the time. The Hamiltonian for this system is

$$H = \frac{1}{2} (\dot{A}_i{}^a)^2 + \frac{g^2}{4} \left[(A_i{}^a A_i{}^a)^2 - (A_i{}^a A_j{}^a)^2 \right]. \quad (10)$$

Following Ref. [4], one solution of Eq. (10) can be expressed as the following nine-parameter form

$$A_i{}^a = \frac{1}{g} O_i{}^a f^{(a)}(t), \quad (11)$$

where there is no summation over a , O_i^a is a constant orthogonal matrix i.e. $O_i^a O_i^b = \delta^{ab}$, and $f^{(a)}$ denotes the three colors. Now we find

$$\ddot{f}^{(a)} + f^{(a)} \left(\mathbf{f}^2 - f^{(a)2} \right) = 0, \quad (12)$$

where $\mathbf{f}^2 = f^{(1)2} + f^{(2)2} + f^{(3)2}$. For the case $f^{(3)} = 0$, if we introduce $x = f^{(1)}$ and $y = f^{(2)}$, the nonlinear equations of motion are

$$\ddot{x} + xy^2 = 0, \quad \ddot{y} + yx^2 = 0. \quad (13)$$

So, the corresponding Hamiltonian is given by

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}p_y^2 + \frac{1}{2}x^2y^2, \quad (14)$$

which is well-known in the classical and quantum chaos studies [11–16]. The quartic potential $V(x, y) = x^2y^2$ appears in various branches of science such as chemistry, astrophysics, and cosmology. In particular, in the classical scalar electrodynamics without self-interaction of the scalar field, when only a single component of the electromagnetic gauge field is nonvanishing, it is classically equivalent to the two-dimensional dynamical system with the potential x^2y^2 that shows a strong chaotic behavior [17]. This potential also appears in the homogeneous limit of the Yang-Mills equations [18, 19].

4 The optimized trigonometric basis-set expansion method

In the quantum domain, the Hamiltonian (14) leads to the following two-dimensional time-independent Schrödinger equation ($\hbar = 1$)

$$-\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x, y) + V(x, y)\psi(x, y) = \epsilon \psi(x, y), \quad (15)$$

where $V(x, y) = x^2y^2$, $\epsilon = 2E$, E is the energy of the system, i.e. $H\psi(x, y) = E\psi(x, y)$, and the wave function $\psi(x, y)$ usually satisfies the following condition

$$\lim_{x, y \rightarrow \infty} \psi(x, y) = 0. \quad (16)$$

To find the approximate energy eigenvalues, we implement a two-dimensional Rayleigh-Ritz variational method whereas the solutions are determined upon two independent variables, namely the truncated domains in x and y directions. The important point of the method is the consideration of a truncated domain of the independent variables x and y so that

$$x \in \left[-\frac{L_x}{2}, +\frac{L_x}{2}\right], \quad y \in \left[-\frac{L_y}{2}, +\frac{L_y}{2}\right], \quad (17)$$

and the modification of the usual boundary condition (16). Thus, the problem is finding the solution of $H\psi = E\psi$ subject to the Dirichlet boundary conditions

$$\psi\left(-\frac{L_x}{2}, y\right) = \psi\left(+\frac{L_x}{2}, y\right) = \psi\left(x, -\frac{L_y}{2}\right) = \psi\left(x, +\frac{L_y}{2}\right) = 0, \quad (18)$$

for all values of x and y on the boundaries of the finite rectangular region. This method gives highly accurate results if both the truncated domain and the number of the basis functions are adjusted properly in one [6, 7, 9] or two [10] dimensions.

To proceed further, we choose a finite set of the trigonometric basis functions obeying Dirichlet boundary condition. Moreover, to simplify the diagonalization procedure, we shift the domain to $0 < x < L_x$ and $0 < y < L_y$ and write

$$\psi(x, y) = \sum_{m,l=1}^{\infty} A_{ml} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{l\pi y}{L_y}\right). \quad (19)$$

Now since we can write $V(x, y)\psi(x, y) = \sum_{m,l} B_{ml} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{l\pi y}{L_y}\right)$, we find

$$\left[\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{l\pi}{L_y}\right)^2\right] A_{ml} + B_{ml} = \epsilon A_{ml}. \quad (20)$$

The matrix B is determined by $B_{m,l} = \sum_{m',l'} C_{mm'l'l'} A_{m'l'}$ where

$$C_{mm'l'l'} = \frac{4}{L_x L_y} \int_0^{L_x} \int_0^{L_y} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{l\pi y}{L_y}\right) V(x, y) \sin\left(\frac{m'\pi x}{L_x}\right) \sin\left(\frac{l'\pi y}{L_y}\right) dx dy. \quad (21)$$

Therefore, Eq. (20) can be rewritten as

$$\left[\left(\frac{m\pi}{L_x}\right)^2 + \left(\frac{l\pi}{L_y}\right)^2\right] A_{ml} + \sum_{m',l'} C_{mm'l'l'} A_{m'l'} = \epsilon A_{ml}. \quad (22)$$

Notice that the presence of the potential term leads to nonzero coefficients $C_{m,m',l,l'}$ in Eq. (22), which couples all of the matrix elements of A . To diagonalize the Hamiltonian, we select the first N^2 basis functions by letting the indices m and l run from 1 to N . For this case, we replace Eq. (19) with the expansion of the solutions in terms of $N \times N$ basis functions in two-dimensional space, namely $\psi(x, y) = \sum_{m,l=1}^N A_{ml} \sin\left(\frac{m\pi x}{L_x}\right) \sin\left(\frac{l\pi y}{L_y}\right)$. Then we replace the square matrix A with a column vector \tilde{A} with N^2 elements, so that any element of A corresponds to one element of \tilde{A} . With this replacement Eq. (22) can be written as

$$D\tilde{A} = \epsilon\tilde{A}, \quad (23)$$

where D is a square matrix with $N^2 \times N^2$ elements and can be obtained from Eq. (22). So the solution to this matrix equation simultaneously yields N^2 sought after energy eigenvalues and eigenstates, namely $\psi_n(x, y)$ where $n = 1, 2, \dots, N^2$. Now the optimization procedure is the adjustment of L_x and L_y for each N and we denote these optimal quantities by $\hat{L}_x(N)$ and $\hat{L}_y(N)$ which correspond to the minimum value of $\epsilon(N, L_x, L_y)$ for a fixed N . Indeed, highly accurate solutions can be obtained upon using these optimal lengths.

For our case, since the potential contains the interchange symmetry, i.e., $V(x, y) = V(y, x)$ we solve the problem on a square domain putting $L_x = L_y$ [10]. Note that we only imposed the condition of the vanishing of the wave functions at the boundaries. Physically, it means that the potential is infinite outside. Indeed, the potential is $V(x, y) = x^2 y^2$ inside and $V(x, y) = \infty$ outside. This situation is similar to the particle in a box where the potential inside does not vanish anymore. This approximation is valid for the low-lying energy eigenvalues where their corresponding eigenstates almost vanish at the boundaries.

In Table 1, we have shown the eigenvalues for the first 12 states and some highly excited ones, namely, $n = \{21, 26, 34, 45\}$. These highly excited states are chosen for their high accuracy due to their symmetric form, and the fact that we have chosen $\hat{L}_x(N) = \hat{L}_y(N) = \hat{L}(N)$. Figure 1 shows the ground state wave function. Notice the slight over extension of the wave function in the x and y directions due to the

(m, l)	n	ϵ_n	error
(1,1)	1	1.10822315780256	1.19×10^{-10}
(1,2)	2	2.37863785124994	1.16×10^{-8}
(2,1)	3	2.37863785124996	1.16×10^{-8}
(2,2)	4	3.05608156130323	2.06×10^{-7}
(3,3)	5	3.51495134040797	1.12×10^{-6}
(2,3)	6	4.09348955687600	9.38×10^{-6}
(3,2)	7	4.09348955687604	9.38×10^{-6}
(4,4)	8	4.75298944936096	9.32×10^{-5}
(5,5)	9	4.98538290136962	1.75×10^{-5}
(6,6)	10	5.01127928161308	4.59×10^{-11}
(1,3)	11	5.50103621623983	7.92×10^{-4}
(3,1)	12	5.50103621623990	7.92×10^{-4}
(11,11)	21	8.07437393671447	6.64×10^{-9}
(14,14)	26	9.27305945794927	3.36×10^{-8}
(18,18)	34	11.4718771513251	7.24×10^{-7}
(23,23)	45	13.8662683175987	8.33×10^{-6}
	$\hat{L}(42)$	15.53	

Table 1: The first 12 states and some highly excited states with $N = 42$ basis functions.

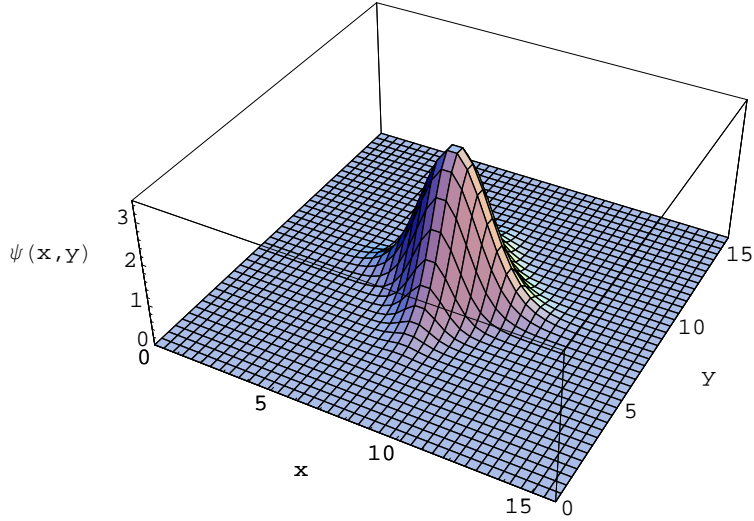


Figure 1: The ground state wave function using $N = 42$ and $\hat{L}(42) = 15.53$.

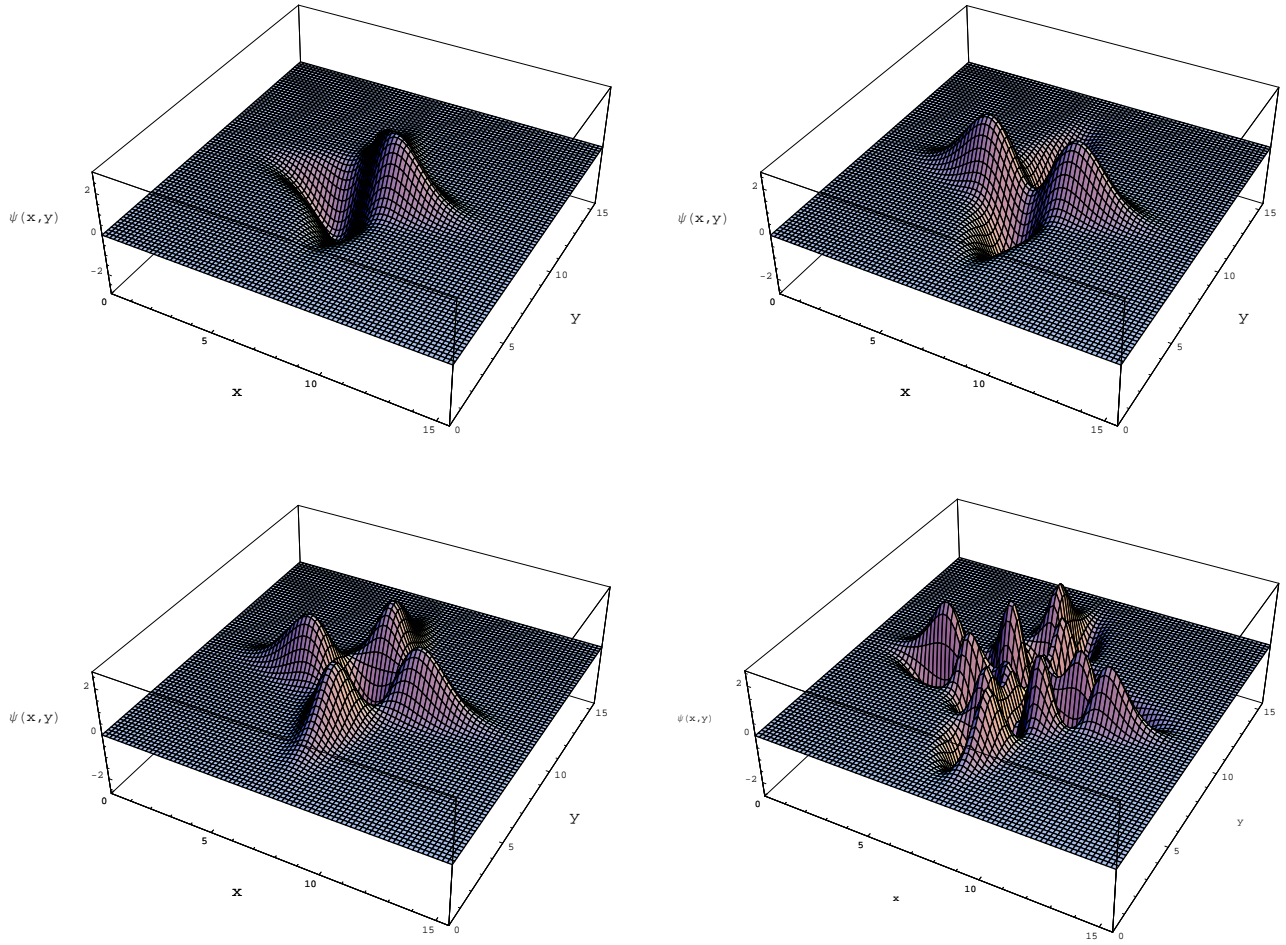


Figure 2: Upper left: 2nd state, upper right: 4th state, lower left: 5th state, and lower right: 44th state.

particular form of the potential. In Fig. 2 we have shown the wave functions for the second, forth, fifth, and forty forth eigenstates. The third eigenstate is degenerate with the second state and can be obtained from it by 90 degree rotation. Note that, as it can be seen from Eqs. (21) and (22), exchanging quantum numbers m and l leads to the same eigenvalue equation. Thus, energy levels with exchanged quantum numbers should be degenerate. In particular, the values of (m, l) which are shown in Table 1 explicitly exhibit the degeneracy of the problem.

5 Color space with one degree of freedom

In this section we are interested to study the particular solution of Eq. (12), namely $f^{(1)} = f^{(2)} = f^{(3)} = f$ [5]. For this case the equation of motion is

$$\ddot{f} + 2f^3 = 0. \quad (24)$$

So the solution is

$$f(t) = \left(\frac{2g^2}{3}\right)^{1/4} \mu \operatorname{cn} \left[\left(\frac{8g^2}{3}\right)^{1/4} \mu(t + t_0); \frac{1}{\sqrt{2}} \right], \quad (25)$$

where $\operatorname{cn}(x; k)$ is the Jacobian elliptic cosine of argument x and modulus k , μ^4 is the energy density T_{00} in the present coordinate system, and t_0 is the arbitrary origin of the time. So the solution is periodic with period $T = (3/8g^2)^{1/4} (4/\mu) K(1/\sqrt{2})$ where $K(x)$ is the complete elliptic integral of the first kind. Now, if we take $x = f$ the corresponding Hamiltonian reads

$$H = \frac{1}{2}p_x^2 + \frac{1}{2}x^4. \quad (26)$$

Note that the quartic potential has attracted much attention in literature because of its similarity to the $\lambda\phi^4$ quantum field theory as the prototype of spontaneous symmetry breaking.

In the quantum domain the above Hamiltonian results in the following one-dimensional Schrödinger equation

$$-\frac{\partial^2 \psi(x)}{\partial x^2} + x^4 \psi(x) = \epsilon \psi(x), \quad (27)$$

n	$\epsilon_n^{\text{exact}}$	WKB
1	0.530181045	0.4335
3	3.727848969	3.7069
5	8.130913009	8.1168
7	13.26423559	13.253
9	18.96150051	18.952
11	25.12812726	25.120
13	31.70152349	31.694
15	38.63660024	38.630
17	45.89903340	45.893
19	53.46165369	53.456
21	61.30231950	61.297

Table 2: The low-lying energy eigenvalues for the one-dimensional anharmonic oscillator (26) using highly accurate numerical and semiclassical schemes.

where $\epsilon = 2E$. This equation is the Schrödinger equation for the anharmonic oscillator in one-dimension and is not exactly solvable. However, we can use the optimized trigonometric basis-set expansion method to find the highly accurate solutions [6–9]. On the other hand, the semiclassical approximation ($\hbar = 1$)

$$\oint p_x dx = 2\pi \left(n - \frac{1}{2} \right), \quad n = 1, 2, \dots, \quad (28)$$

results in

$$E_n = \frac{1}{2} \left[\frac{\sqrt{\pi} \Gamma(7/4)}{\Gamma(5/4)} \left(n - \frac{1}{2} \right) \right]^{4/3}. \quad (29)$$

In Table 2 we have reported the low-lying energy eigenvalues for the anharmonic oscillator using the optimized trigonometric basis-set expansion and the semiclassical methods. As it can be seen from the table the agreement between the exact and semiclassical results increases for the excited states with large quantum numbers.

6 Conclusions

Since the Yang-Mills equations are nonlinear, they are not explicitly solvable in general. This is similar to the Einstein equations for the gravitational field but unlike the Maxwell equations for the electromagnetic field. However, like the Maxwell equations they describe massless waves that travel at the speed of light at the classical level. Although the classical non-abelian gauge theory is within the reach of

established mathematical methods, the precise definition of quantum gauge theory in four dimensions is still unclear. In this paper and in the context of the first quantization, we have studied the quantum Yang-Mills mechanics in the absence of external sources corresponding to the group $SU(2)$. In the classical domain, this system in a coordinate system in which the Poynting vector vanishes, exhibited nonlinear oscillations of the color degrees of freedom. We solved its corresponding Schrödinger equations in quantum domain in which the potential term becomes x^4 and x^2y^2 in one and two dimensions, respectively. Using the optimized trigonometric basis expansion method we found the accurate energy eigenvalues and eigenstates and compared the results with the semiclassical solutions.

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